Unique Paper Code : **12271102**

Name of the Paper : **Mathematical Methods for Economics I**

Name of the Course: **B.A. (Hons.) Economics**

Semester : **I**

Duration: 3 Hours

Maximum Marks: 75

Instructions for the candidates:

1. Answers may be written either in English or in Hindi; but the same medium should be used throughout the paper.
2. **There are six questions in all. Attempt any four.**
3. All parts of a question must be answered together.
4. All questions carry equal (18.75) marks.
5. Use of a simple calculator is allowed.

1. (a) A function f is given by

$$f(x) = \left(1 + \frac{3}{x}\right)\sqrt{x - 7}$$

(i) Find the domain of f, the zeroes of f, and the interval where f is positive.

(ii) Find the possible local extreme points and values.

(iii) Examine $f(x)$ as $x \to 0^-$, $x \to 0^+$, $x \to \infty$. Also, determine the limit of $f'(x)$ as $x \to \infty$. Does f has a maximum or a minimum in the domain?

(b) Use the first four terms of the binomial expansion of $\left(1 - \frac{1}{50}\right)^{\frac{1}{2}}$ to derive the approximation $\sqrt{2} \approx 1.414214$

(c) Show that following matrix A is invertible, and find the inverse A^{-1}

$$A = \begin{pmatrix} 3 & 2 & 5 \\ 4 & 3 & 2 \\ 5 & 2 & 3 \end{pmatrix}$$

(d) Suppose you inherit a piece of land whose market value t years from now is estimated to be $V(t) = 50000e^{\sqrt{2t}}$. If the prevailing rate of interest remains constant at 10%, when will it be most advantageous for you to sell the land?

(e) Find the equation for the plane through the points $(3,4,-3)$, $(5,2,1)$, and $(2,-1,4)$.

(5, 4, 3, 3, 3.75)

2. (a) Consider the following system of equations:

$$2yz + zx - 5xy = 2$$

$$yz - zx + 2xy = 1$$

$$yz - 2zx + 6xy = 3$$

Show that $xyz = \pm 6$. And find all the possible values of $x, y,$ and z.

(b) Sketch the following subsets of the $x - y$ plane

(i) $|x - 1| + |y - 1| \leq 1$

(ii) $|x||y - 2| \leq 1$

(c) For the following function, find the expression for $\frac{dy}{dx}, \frac{dx}{dy}$

$$y = \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}$$

(d) Find all the solutions of the equation

$$|x + 1| - |x| + 3|x - 1| - 2|x - 2| = x + 2$$

(e) Determine if the function $f(x) = 2^{ln\sqrt{3x+4}}$ is concave or convex. Does it have a global maximum, minimum, point of inflection?

(5, 4, 3, 3, 3.75)

3. (a) Consider the function f given by

$$f(x) = x^{\frac{2}{3}}(6 - x)^{\frac{1}{3}}$$

Find the interval where f is increasing, the interval where f is decreasing, points of maximum, and
points of minimum. And plot the graph.

(b) For the following function, find $f'(x)$

$$f(x) = 3\left(\frac{3^x 7^x}{3^x + 4(7)^x}\right)^{\frac{5}{x}}$$

(c) Consider a monopolist who sells x units of Beans in Yemen. The price received is given by $P(x) = d - ex$ (where d and e are positive constants). His total cost is given by $C(x) = ax^2 + bx + c$ (where $a, b,$ and c are positive constants). Find the profit maximising output of Beans. Suppose the government imposes a tax on Beans of t per unit. Find an expression for the monopolist's profit and the new quantity shipped. Calculate the government's tax revenue as a function of t, and find the revenue maximising tax rate.

(d) Determine the rank of the following matrix K, for all values of p:

$$K = \begin{pmatrix} 8-p & -2 & -4 \\ 2 & 2-p & 0 \\ 1 & 0 & 2-p \end{pmatrix}$$

(e) Let $g(x) = f(x) + f(1-x)$ and $f''(x) > 0; x \in (0,1)$. Find the intervals of increase and decrease of $g(x)$.

(5, 3, 3.75, 5, 2)

4. (a) A study of paper machines in Industrial production in India from 1990 onwards estimated that the number z in use (measured in lakhs), as a function of time t (measured in years), so that $t = 0$ corresponds to 1990, is given by

$$z = 250.9 + \frac{228.46}{1+8.11625e^{-0.340416t}}$$

(i) Find the number of paper machines in 1990. How many machines were added in the decade up to 2000?

(ii) Find the limit for z as $t \to \infty$, and draw the graph.

(b) The equation $f(e^x) - g(x+y) = h(ln(y))$ defines y as an implicit function of x, for $x \in \mathbb{R}$ and $y > 0$. Find $\frac{dy}{dx}$ and determine its sign if $f' < 0, g' < 0$ and $h' > 0$.

(c) Suppose A and B are $n \times n$ matrices such that $AB = B^3A$. Prove that $(AB)^2 = B^{12}A^2$.

(d) Test the convergence of the sequence $S_n = \left\{(-1)^n \frac{2n^3}{n^3+1}\right\}_{n=1}^{\infty}$

(e) Is the following function continuous at $x = 0$

$$f(x) = \begin{cases} \dfrac{x\left(e^{\frac{-1}{x}} - e^{\frac{1}{x}}\right)}{e^{\frac{-1}{x}} + e^{\frac{1}{x}}}, x \neq 0 \\ 0, x = 0 \end{cases}$$

Is it differentiable at $x = 0$?

(f) Solve the following system of linear equations:

$$4x + 5y + 6z = 23$$
$$3x + 6y + 4z = 21$$
$$2x + 7y + 4z = 18$$

(5, 3, 3, 2, 3, 2.75)

5. (a) The line L is given by $x_1 = -t + 2, x_2 = 2t - 1$, and $x_3 = t + 3$

   (i) Verify that the point $a = (2, -1, 3)$ lies on L, but that $(1,1,1)$ doesn't.

   (ii) Determine the direction of L.

   (iii) Find the equation of the plane through a that is orthogonal to L.

   (iv) Find the point where L intersects the plane $3x_1 + 5x_2 - x_3 = 6$.

   (b) Find the intervals where the following cost function $C(x)$ is convex and where it is concave, find the unique inflection point:

$$C(x) = ax^3 + bx^2 + cx + d, (a > 0, b < 0, c > 0, d < 0)$$

   (c) Show that if f and g are functions for which $f'(x) = g(x)$ and $g'(x) = f(x)$, then $f^2(x) - g^2(x)$ is a constant.

   (d) Find the inverse of the function $f(x) = -x^6 + 5, x > 0$ (if it exists). Also, find the global maxima.

   (e) Solve for all possible real values of$x$satisfying:

$$x^6 + 9x^3 + 8 = 0$$

   (f) Consider the following two statements A and B

   A: number n is odd,

   B: n is a prime number strictly greater than 2.

   Check whether A is necessary or sufficient or both necessary and sufficient condition for B.

(6, 4, 3, 2, 2, 1.75)

6. (a) Suppose that the price of a precious metal after x years is given by $P(x) = Me^n$, where M and n are constants.

   (i) Find M and n when $P(0) = 4$ and $P'(0) = 1$. In this case, what is the price after 10 years?

   (ii) Assuming calculated values of M and n from (i). When the price has increased to 18, it becomes controlled so that the annual price increase is limited to 10%. When are price controls first needed? What length of time is needed for the price to double before and after price controls are introduced?

(b) Prove the following inequalities

(i) $ln\left(\frac{1+x}{1-x}\right) > 2x$ for $0 < x < 1$

(ii) $e^x > 1 + x + \frac{x^2}{2}$, for $x > 0$

(c) Find a and b, such that the following function, f has vertical asymptote at $x = 5$, and horizontal asymptote at $y = -3$.

$$f(x) = \frac{ax+5}{3-bx}$$

(d) Consider the following system of linear equations:

$$2x - y + 3z = 2$$

$$x + y + 2z = 2$$

$$5x - y + pz = q$$

(i) For what real values of p and q, the following system of linear equations have infinitely many solutions?

(ii) For the value of $p = 3$ and $q = 5$, find all possible solutions to the system.

(e) Find the points of maximum, and the points of minimum for the following function $f(x)$, in the interval $[0,1]$

$$f(x) = \frac{1}{x(1-x)}$$

(5, 3, 3, 6, 1.75)